On imbedding theorems

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Dedicated to Professor B. Szőkefalvi-Nagy on his 60th birthday

Let $\varphi(x)\equiv\varphi_p(x)$ $(p\geqq 1)$ be a nonnegative increasing function on $[0,\infty)$ with the following properties:

(1) $$\frac{\varphi(x)}{x}\uparrow \quad\text{and}\quad \frac{\varphi(x)}{x^p}\downarrow \quad\text{as}\quad x\to\infty.$$

The set of the measurable functions $f(x)$ on $[0, 1]$ for which $\int_0^1 \varphi(|f(x)|)\,dx<\infty$ will be denoted by $\varphi(L)$.

If $f(x)\in\varphi(L)$ then the "modulus of continuity of $f(x)$ with respect to φ" will be defined by

$$w_\varphi(\delta,f)=\sup_{0\leqq h\leqq\delta}\bar\varphi\Big(\int_0^{1-h}\varphi(|f(x+h)-f(x)|)\,dx \qquad (0\leqq\delta\leqq 1),$$

where $\bar\varphi(x)$ denotes the inverse function of $\varphi(x)$.

If $\varphi(x)=x^p$ $(p\geqq 1)$ then $\varphi(L)$ and $\omega_\varphi(\delta,f)$ will be denoted, as usual, by L^p and $\omega_p(\delta,f)$, respectively.

P. L. Ul'janov has proved imbedding theorems in several papers (see for instance [4] and [5]). Among other things, he gave conditions which assure that a function $f(x)\in L^p$ $(p\geqq 1)$ should belong to another space L^ν $(\nu>p)$. A sample theorem is as follows (see [4], Theorem 1): If $f(x)\in L^p$ $(p\geqq 1)$ and

(2) $$\sum_{n=1}^{\infty} n^{\frac{\nu}{p}-2}\,\omega_p^\nu\left(\frac{1}{n},f\right)<\infty,$$

then $f(x)\in L^\nu$.

In [1] we generalized some of Ul'janov's results and gave conditions assuring the transition from L^p to $L^p(\ln^+ L)^\beta$ (see Corollary 1, in case $p=1$ see also Theorem 2 in [4]) and from L^p to $\varphi_\nu(L)$ (see Theorem 2). The latter result states that if $f(x)\in L^p$ and

(3) $$\sum_{n=1}^{\infty}\frac{1}{n^2}\varphi_\nu\left(n^{1/p}\omega_p\left(\frac{1}{n}\right)\right)<\infty$$

then $f(x)\in\varphi_\nu(L)$.

It is clear that in the special case $\varphi_\nu(x)=x^\nu$ (3) reduces to (2).

In the present paper we are going to give conditions which assure the transition from an arbitrary collection $\varphi_p(L)$ to another $\varphi_\nu(L)$.

More precisely we prove the following theorems

Theorem 1. *Let $f(x)\in\varphi(L)$ $\left(\varphi(x)\equiv\varphi_p(x),\, p\geqq 1\right)$ and let $\{\lambda_k\}$ be a nonnegative monotonic sequence of numbers such that*

$$\sum_{k=m}^{\infty}\frac{\lambda_k}{k^{1+\varepsilon}}\leqq K(\lambda)\frac{\lambda_m}{m^\varepsilon},\ ^{1)} \tag{4}$$

where $\varepsilon=(4[p+1]+2)^{-1}$, [2] *and furthermore let $\Lambda(x)=\sum_{k=1}^{x}\frac{\lambda_k}{k}$.* [3]

Then $\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right)<\infty$ implies $f(x)\in\varphi(L)\Lambda(L)$, and

$$\int_0^1\varphi(|f(x)|)\Lambda(|f(x)|)\,dx\leqq K(\varphi,\lambda)\left\{\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right)+\int_0^1\varphi(|f(x)|)\,dx\right\}. \tag{5}$$

Theorem 2. *Set $\varphi(x)\equiv\varphi_p(x)$ and $\psi(x)\equiv\varphi_\nu(x)$ $(p,\,\nu\geqq 1)$. Let $\{\varrho_k\}$ be a non-negative nondecreasing sequence of numbers with*

$$\sum_{k=m}^{\infty}\frac{\varrho_k}{k^2}\leqq K(\varrho)\frac{\varrho_m}{m}, \tag{6}$$

and denote by $\varrho(x)$ the continuous function which is linear between n and $n+1$, furthermore $\varrho(n)=\varrho_n$. Suppose that $f(x)\in\varphi(L)$ and

$$\sum_{n=1}^{\infty}\frac{\varrho_n}{n^2}\psi\left(\bar\varphi\left(n\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right)\right)\right)<\infty.$$

Then $f(x)\in\psi(L)\varrho(L)$, and

$$\int_0^1\psi(|f(x)|)\varrho(|f(x)|)\,dx\leqq$$

$$\leqq K(\varphi,\psi,\varrho)\left\{\sum_{n=1}^{\infty}\frac{\varrho_n}{n^2}\psi\left(\bar\varphi\left(n\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right)\right)\right)+\psi\left(\int_0^1\varphi(|f(x)|)\,dx\right)\right\}. \tag{7}$$

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[1]) K and K_i, denote either absolute constants or constants depending on certain functions and numbers which are not necessary to specify; $K(\alpha,\beta,\ldots)$ and $K_i(\alpha,\beta,\ldots)$ denote positive constants depending only on the indicated parameters. These constants are not necessarily the same at each occurrence.

[2]) $[y]$ denotes the integral part of y.

[3]) $\sum_a^b$, where a and b are not necessarily integers, means a sum over all integers between a and b.

The methods of proof of these theorems are similar to those of Theorem 1 and 2 given in [1], but new lemmas, using the modulus of continuity $\omega_\varphi\left(\frac{1}{n}, f\right)$ instead of $\omega_p\left(\frac{1}{n}, f\right)$, have to be introduced. The proofs of these lemmas run similarly to those of the old ones which were proved partly by P. L. UL'JANOV [4] and me [1].

These theorems have the following two corollaries, which have so far been proved only for $\varphi(x)\equiv x^p$ and $\varrho_k\equiv 1$ (see in [1] Corollary 1 and Theorem 2).

Corollary 1. *If $f(x)\in\varphi(L)$, $\beta>-1$ and*

$$\sum_{n=2}^{\infty}\frac{1}{n}(\ln n)^\beta \varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right)<\infty,$$

then $f(x)\in\varphi(L)(\ln^+ L)^{\beta+1}$.

Corollary 2. *If $f(x)\in L^p$ $(p\geqq 1)$ and*

$$\sum_{n=1}^{\infty}\frac{\varrho_n}{n^2}\psi\left(n^{1/p}\omega_p\left(\frac{1}{n}, f\right)\right)<\infty,$$

then $f(x)\in\psi(L)\varrho(L)$, where $\psi(x)$, ϱ_k and $\varrho(x)$ have the same meanings as in Theorem 2.

From Corollary 2, by choosing special ϱ_k and $\psi(x)$, we obtain two other corollaries.

Corollary 3. *If $f(x)\in L^p$ $(p\geqq 1)$, $0\leqq\alpha<1$, $\beta\geqq 0$ and*

$$\sum_{n=2}^{\infty}\frac{(\ln n)^\beta}{n^{2-\alpha}}\psi\left(n^{1/p}\omega_p\left(\frac{1}{n}, f\right)\right)<\infty,$$

then $f(x)\in\psi(L)L^\alpha(\ln^+ L)^\beta$.

Corollary 4. *If $f(x)\in L^p$ $(p\geqq 1)$, $\nu\geqq p$, $0\leqq\alpha<1$, $\beta\geqq 0$ and*

$$\sum_{n=2}^{\infty}(\ln n)^\beta n^{\frac{\nu}{p}+\alpha-2}\omega_p^\nu\left(\frac{1}{n}, f\right)<\infty,$$

then $f(x)\in L^{\nu+\alpha}(\ln^+ L)^\beta$.

§ 1. Lemmas

We require the following lemmas.

Lemma 1. ([4], Lemma 7'.) *If* $f(x)\in L(0,1)$ *and* $F(z)$ *is a nonnegative nonincreasing function equidistributed with* $|f(x)|$, *that is,*

$$\operatorname{mes}\{x: x\in[0,1], |f(x)|>y\} = \operatorname{mes}\{z: z\in[0,1], F(z)>y\},$$

then

$$\sup_{\substack{E\subset[0,1]\\ |E|=\alpha}} \int_E |f(x)|\,dx = \int_0^\alpha F(z)\,dz$$

for any $0\leqq\alpha\leqq 1$; *furthermore if* $0\leqq\alpha\leqq\frac{1}{2}$ *and*

$$\sup_{\substack{E\subset[0,1]\\ |E|=\alpha}} \int_E |f(x)|\,dx = \int_{E_0} |f(x)|\,dx,$$

then

$$\sup_{\substack{E\subset[0,1]-E_0\\ |E|=\alpha}} \int_E |f(x)|\,dx = \int_\alpha^{2\alpha} F(z)\,dz.$$

Lemma 2. ([5], Remark 1.) *If* $f(x)\in\varphi(L)$ *and* $0\leqq a<b\leqq 1$, *then*

$$\int_a^b\int_a^b \varphi(|f(x)-f(y)|)\,dx\,dy = 2\int_0^{b-a}\Big\{\int_a^{b-u}\varphi(|f(u+y)-f(y)|)\,dy\Big\}\,du.$$

Lemma 3. *Let* $\varphi(x)\equiv\varphi_p(x)$. *If* $u(x)$ *and* $v(x)$ *are nonnegative measurable functions on the interval* I, *then we have*

$$\varphi\left(\frac{\int_I u(x)\,v(x)\,dx}{\int_I u(x)\,dx}\right) \leqq 2^p\,\frac{\int_I u(x)\,\varphi(v(x))\,dx}{\int_I u(x)\,dx}.$$

This lemma immediately follows from results of H. P. Mulholland [2] (see Theorem 1 and Remark 2. 34).

Lemma 4. *If* $a_n\geqq 0$ *and* $\lambda_n>0$, *then*

$$\sum_{n=1}^\infty \lambda_n\varphi\left(\sum_{i=1}^n a_i\right) \leqq K(\varphi)\sum_{n=1}^\infty \lambda_n\varphi\left(\frac{a_n}{\lambda_n}\sum_{k=n}^\infty \lambda_k\right).$$

This lemma is a part of Theorem of J. Németh [3] (see the inequality (8) of Theorem).

Lemma 5. *If $f(x)\in\varphi(L)$ and*

$$\psi_n(x)=n\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt \quad \text{for} \quad x\in\left[\frac{k}{n},\frac{k+1}{n}\right), \qquad 0\leqq k<n,$$

then

$$\int\limits_0^1 \varphi\left(|f(t)-\psi_n(t)|\right)dt \leqq K(\varphi)\,\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right]. \tag{1.1}$$

Proof. Using Lemma 3, we obtain that

$$\int\limits_0^1 \varphi\left(|f(t)-\psi_n(t)|\right)dt=\sum_{k=0}^{n-1}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi\left(\left|f(t)-n\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx\right|\right)dt\leqq$$

$$\leqq \sum_{k=0}^{n-1}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi\left(n\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t)-f(x)|\,dx\right)dt\leqq \tag{1.2}$$

$$\leqq K_1(\varphi)\sum_{k=0}^{n-1}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \left(n\int\limits_{\frac{k}{n}}^{\frac{n+1}{n}} \varphi\left(|f(t)-f(x)|\right)dx\right)dt\equiv I_1.$$

Next we use Lemma 2 and have

$$I_1\leqq 2K_1 n\sum_{k=0}^{n-1}\int\limits_0^{\frac{1}{n}}\left\{\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}-u} \varphi\left(|f(u+y)-f(y)|\right)dy\right\}du\leqq$$

$$\leqq K_2 n\int\limits_0^{\frac{1}{n}}\left\{\int\limits_0^{1-u} \varphi\left(|f(u+y)-f(y)|\right)dy\right\}du\leqq K_2\,\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right].$$

From this and (1.2) we obtain (1.1).

Lemma 6. *If $f(x)\in\varphi(L)$ then*

$$\omega_\varphi\left(\frac{1}{n},f\right)\geqq K(\varphi)\,\bar\varphi\left[\frac{1}{n}\,\varphi\left(n\left(\int\limits_0^{\frac{1}{n}}F(z)\,dz-\int\limits_{\frac{1}{n}}^{\frac{2}{n}}F(z)\,dz\right)\right)\right],$$

where $\bar\varphi(x)$ denotes the inverse function of $\varphi(x)$ and $F(z)$ has the same meaning as in Lemma 1.

Proof. Set

$$\alpha(t) \equiv \alpha_n(t) = n \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} |f(u)|\,du = a_k \geqq 0 \quad \text{for} \quad t \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$$

$(k = 0, 1, \ldots, n-1)$. Denote by $0 \leqq b_0 \leqq b_1 \leqq \cdots \leqq b_{n-1}$ the nondecreasing rearrangement of the sequence $\{a_k\}_0^{n-1}$. Then

$$\int\limits_0^{1-\frac{1}{n}} \varphi\left(\left|\alpha\left(t+\frac{1}{n}\right)-\alpha(t)\right|\right) dt = \sum_{k=0}^{n-2} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi\left(\left|\alpha\left(t+\frac{1}{n}\right)-\alpha(t)\right|\right) dt \geqq \tag{1.3}$$
$$\geqq \sum_{k=0}^{n-2} \varphi(|a_{k+1}-a_k|)\frac{1}{n} \geqq \frac{1}{n}\varphi(b_{n-1}-b_{n-2}).$$

Set $B=\{t : t\in[0,1],\ \alpha(t)=b_{n-1}\}$. Then it is clear that for arbitrary sets $B_1 \subset B$ with $|B_1| = \frac{1}{n}$ and $E \subset [0,1]$ with $|E| = \frac{1}{n}$ we have

$$b_{n-1}-b_{n-2} = n\left(\int\limits_{B_1} \alpha(t)\,dt - \frac{b_{n-2}}{n}\right) \geqq n\Big(\int\limits_E \alpha(t)\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int\limits_{E_1} \alpha(t)\,dt\Big),$$

and thus

$$b_{n-1}-b_{n-2} \geqq n \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}} \Big\{\int\limits_E \alpha(t)\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int\limits_{E_1} \alpha(t)\,dt\Big\}, \tag{1.4}$$

Since $\varphi(|a|) \leqq \varphi(|a+b|+|b|) \leqq K(\varphi)\{\varphi(|a+b|)+\varphi(|b|)\}$, we obatin

$$\varphi(|a+b|) \geqq K_1\varphi(|a|) - \varphi(|b|). \tag{1.5}$$

Using (1.3), (1.4) and (1.5) we have

$$\varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right) \geqq \varphi\left(\omega_\varphi\left(\frac{1}{n}, |f|\right)\right) \geqq$$
$$\geqq \int\limits_0^{1-\frac{1}{n}} \varphi\left(\left|\left|f\left(t+\frac{1}{n}\right)\right| - \alpha\left(t+\frac{1}{n}\right) + \alpha\left(t+\frac{1}{n}\right) - \alpha(t) + \alpha(t) - |f(t)|\right|\right) dt \geqq$$
$$\geqq K_1 \int\limits_0^{1-\frac{1}{n}} \varphi\left(\left|\alpha\left(t+\frac{1}{n}\right)-\alpha(t)\right|\right) dt - K_2 \int\limits_0^1 \varphi(||f(t)|-\alpha(t)|)\,dt \geqq$$
$$\geqq \frac{K_1}{n}\varphi\Big(n \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}} \Big\{\int\limits_E \alpha(t)\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int\limits_{E_1} \alpha(t)\,dt\Big\}\Big) - K_2\varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right).$$

Hence we obtain that

$$K_3\,\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right) \geqq \frac{1}{n}\,\varphi\Big(n \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}}\Big\{\int_E \alpha(t)\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int_{E_1}\alpha(t)\,dt\Big\},$$

which implies, by $\bar\varphi(ca)\leqq c\bar\varphi(a)$ $(a\geqq 0,\ c\geqq 1)$, that

$$(1.6)\qquad K_4\,\bar\varphi\left(n\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right)\right) \geqq n \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}}\Big\{\int_E \alpha(t)\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int_{E_1}\alpha(t)\,dt\Big\}.$$

An easy computation gives that

$$\sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}}\Big\{\int_E \alpha(t)\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int_{E_1}\alpha(t)\,dt\Big\} \geqq$$

$$\geqq \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}}\Big\{\int_E |f(t)|\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int_{E_1}|f(t)|\,dt\Big\} - 2\sup_{\substack{A\subset[0,1]\\ |A|=\frac{1}{n}}}\int_A \big||f(t)|-\alpha(t)\big|\,dt.$$

This and (1. 6) imply

$$\varphi\Big(n \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}}\Big\{\int_E |f(t)|\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int_{E_1}|f(t)|\,dt\Big\}\Big) \leqq$$

$$\leqq \varphi\left(2n \sup_{\substack{A\subset[0,1]\\ |A|=\frac{1}{n}}}\int_A \big||f(t)|-\alpha(t)\big|\,dt + K_4\,\bar\varphi\left(n\varphi\left(\varphi_\varphi\left(\frac{1}{n},f\right)\right)\right)\right).$$

Consequently, by Lemma 3 and Lemma 5, we have

$$\varphi\Big(n \sup_{\substack{E\subset[0,1]\\ |E|=\frac{1}{n}}}\Big\{\int_E |f(t)|\,dt - \sup_{\substack{E_1\subset[0,1]-E\\ |E_1|=\frac{1}{n}}} \int_{E_1}|f(t)|\,dt\Big\}\Big) \leqq$$

$$\leqq K_5\left\{n \sup_{\substack{A\subset[0,1]\\ |A|=\frac{1}{n}}}\int_A \big(\big||f(t)|-\alpha(t)\big|\big)\,dt + n\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right)\right\} \leqq$$

$$\leqq K_6\, n\varphi\left(\omega_\varphi\left(\frac{1}{n},f\right)\right).$$

Hence, by Lemma 1, the statement of Lemma 6 immediately follows.

Lemma 7. *If $f(x)\in\varphi(L)$ then*

$$F(2^{-n}) \leqq K(\varphi)\Big\{\int_0^1 \varphi(|f(x)|)\,dx + \sum_{k=1}^{n-1} \bar\varphi\big(2^k\varphi(\omega_\varphi(2^{-k},f))\big)\Big\} \tag{1.7}$$

for any $n\geqq 1$, where $F(z)$ has the same meaning as in Lemma 1.

Proof. By Lemma 6 we have

$$K_1\bar\varphi\big(2^k\varphi(\omega_\varphi(2^{-k},f))\big) \geqq 2^k\Big(\int_0^{2^{-k}} F(z)\,dz - \int_{2^{-k}}^{2^{-k+1}} F(z)\,dz\Big)$$

and

$$K_1\bar\varphi\big(2^{k+1}\varphi(\omega_\varphi(2^{-k-1},f))\big) \geqq 2^{k+1}\Big(\int_0^{2^{-k-1}} F(z)\,dz - \int_{2^{-k-1}}^{2^{-k}} F(z)\,dz\Big),$$

whence

$$K_2\bar\varphi\big(2^k\varphi(\omega_\varphi(2^{-k},f))\big) \geqq 2^k\Big(2\int_0^{2^{-k-1}} F(z)\,dz - \int_{2^{-k}}^{2^{-k+1}} F(z)\,dz\Big) \geqq$$

$$\geqq F(2^{-k-1}) - F(2^{-k}).$$

Summing up this inequality from 1 to $n-1$ we obtain (1.7).

Lemma 8. *If $f(x)\in\varphi_p(L)$ and $R=2^{2p+1}$, then*

$$\int_0^{\frac{1}{Rn}} \varphi(F(x))\,dx \leqq \int_{\frac{1}{Rn}}^{\frac{1}{n}} \varphi(F(x))\,dx + K(\varphi)\,\varphi\Big[\omega_\varphi\Big(\frac{1}{n},f\Big)\Big]$$

for any $n\geqq 1$.

Proof. Set

$$E_n^* = \Big\{x:\ x\in[0,1],\ |f(x)| > F\Big(\frac{1}{Rn}\Big)\Big\}$$

and

$$E_n^{**} = \Big\{x:\ x\in[0,1],\ |f(x)| = F\Big(\frac{1}{Rn}\Big)\Big\}.$$

If $|E_n^*| < \dfrac{1}{Rn}$, then define y so that

$$|E_n^{**}\cap(0,y)| = \frac{1}{Rn} - |E^*|,$$

furthermore let

$$E_n = (E_n^{**}\cap(0,y))\cup E_n^*$$

If $|E_n^*| = \dfrac{1}{Rn}$, then set $E_n = E_n^*$.

First we estimate the integral of $\varphi(|f(x)|)$ on E_n. Let $\psi_n(x)$ be the same function as in Lemma 5. By (1) we have

$$\varphi(|f(x)|) \leqq 2^p\{\varphi(|f(x)-\psi_n(x)|)+\varphi(|\psi_n(x)|)\}.$$

Hence

$$\int_{E_n}\varphi(|f(x)|)\,dx \leqq 2^p\Big\{\int_{E_n}\varphi(|f(x)-\psi_n(x)|)\,dx+\int_{E_n}\varphi(|\psi_n(x)|)\,dx\Big\}. \tag{1.8}$$

Since $\varphi(x)$ is increasing on $[0,\infty)$, in view of Lemma 1 we also have

$$\int_{E_n}\varphi(|f(x)|)\,dx=\int_0^{\frac{1}{Rn}}\varphi(F(x))\,dx$$

(see also [6], p. 29), and by Lemma 5

$$\int_0^1\varphi(|f(x)-\psi_n(x)|)\,dx \leqq K(\varphi)\,\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right];$$

consequently (see (1.8))

$$\int_0^{\frac{1}{Rn}}\varphi(F(x))\,dx=K_1\,\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right]+\frac{2^p}{Rn}\varphi(\max|\psi_n(x)|). \tag{1.9}$$

But

$$\max|\psi_n(x)| \leqq n\int_0^{\frac{1}{n}}F(x)\,dx,$$

thus by Lemma 3

$$\varphi(\max|\psi_n(x)|) \leqq 2^p n\int_0^{\frac{1}{n}}\varphi(F(x))\,dx.$$

Therefore, by (1.9) and by the definition of R, we have

$$\int_0^{\frac{1}{Rn}}\varphi(F(x))\,dx \leqq K_1\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right]+\frac{2^{2p}}{R}\int_0^{\frac{1}{n}}\varphi(F(x))\,dx \leqq$$

$$\leqq K_1\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right]+\tfrac{1}{2}\int_0^{\frac{1}{Rn}}\varphi(F(x))\,dx+\tfrac{1}{2}\int_{\frac{1}{Rn}}^{\frac{1}{n}}\varphi(F(x))\,dx,$$

whence the statement of Lemma 8 follows.

Lemma 9. *If* $f(x)\in\varphi(L)$ $(\varphi(x)\equiv\varphi_p(x))$ *and* $\varepsilon=(4[p+1]+2)^{-1}$, *then*

$$\int_0^{\frac{1}{n}}\varphi(F(x))\,dx\leqq\frac{K(\varphi)}{n^{\varepsilon}}\left\{\sum_{k=1}^{n}k^{\varepsilon-1}\varphi\left(\omega_\varphi\left(\frac{1}{k},f\right)\right)+\int_0^1\varphi(F(x))\,dx\right\}$$

for any $n\geqq 1$.

Proof. Set $\mu=2[p+1]+1$. Since $2^{\mu}>R=2^{2p+1}$, by Lemma 8 we have

$$\int_0^{2^{-n}}\varphi(F(x))\,dx=\Big(\int_0^{2^{-\mu-n}}+\int_{2^{-\mu-n}}^{2^{-n}}\Big)\varphi(F(x))\,dx\leqq \tag{1.10}$$

$$\leqq 2\int_{2^{-\mu-n}}^{2^{-n}}\varphi(F(x))\,dx+K\varphi(\omega_\varphi(2^{-n},f)).$$

Furthermore it is clear that

$$\sum_{k=n}^{\infty}\int_{2^{-k-\mu}}^{2^{-n}}\varphi(F(x))\,dx\leqq\mu\sum_{k=n}^{\infty}\int_{2^{-k-1}}^{2^{-k}}\varphi(F(x))\,dx=\mu\int_0^{2^{-n}}\varphi(F(x))\,dx. \tag{1.11}$$

Thus, by (1.10) and (1.11), we get

$$\sum_{k=n}^{\infty}\int_{2^{-k-\mu}}^{2^{-k}}\varphi(F(x))\,dx\leqq 2\mu\Big\{\int_{2^{-\mu-n}}^{2^{-n}}\varphi(F(x))\,dx+K_1\varphi(\omega_\varphi(2^{-n},f))\Big\}. \tag{1.12}$$

Let $N=2\mu$, and, furthermore, let us define for every $n\geqq 1$

$$a_n=\int_{2^{-\mu-n}}^{2^{-n}}\varphi(F(x))\,dx,\qquad b_n=K_1\varphi(\omega_\varphi(2^{-n},f))$$

and

$$\alpha_n=\sum_{k=(n-1)N+1}^{nN}a_k,\qquad \beta_n=\sum_{k=(n-1)N+1}^{nN}b_k.$$

Considering (1.12) we have

$$\sum_{k=n}^{\infty}a_k\leqq N(a_n+b_n). \tag{1.13}$$

Since $a_n\geqq 0$, by (1.13) we obtain that, for any nonnegative integers m and j,

$$\sum_{i=m+1}^{\infty}\alpha_i=\sum_{k=mN+1}^{\infty}a_k\leqq\sum_{k=mN+1-j}^{\infty}a_k=N(a_{mN+1-j}+b_{mN+1-j}).$$

Hence, taking $j=1,2,\ldots,N$ and summing up, we get

$$N\sum_{i=m+1}^{\infty}\alpha_i\leqq N\sum_{j=1}^{N}(a_{mN+1-j}+b_{mN+1-j})=N\sum_{k=(m-1)N+1}^{mN}(a_n+b_k)=N(\alpha_m+\beta_m).$$

Consequently

$$\sum_{i=m+1}^{\infty} \alpha_i \leqq \alpha_m + \beta_m.$$

Multiplying this inequality by $\max(2^{m-2}, 1)$ for all m, $1 \leqq m \leqq n$, by summing and simplifying we obtain

$$2^{n-1} \sum_{i=n+1}^{\infty} \alpha_i \leqq \alpha_1 + \beta_1 + \sum_{k=2}^{n} 2^{k-2} \beta_k. \tag{1.14}$$

Inserting the definitions of α_i and β_i, from (1. 14) it follows that

$$2^{n-1} \sum_{k=nN+1}^{\infty} a_k \leqq \sum_{k=1}^{N} (a_k + b_k) + \sum_{k=2}^{n} 2^{k-2} \sum_{i=(k-1)N+1}^{kN} b_i,$$

that is

$$\sum_{k=nN+1}^{\infty} \int_{2^{-\mu-k}}^{2^{-k}} \varphi(F(x))\,dx \leqq \tag{1.15}$$

$$\leqq 2^{-n}\left\{2 \sum_{k=1}^{N} \int_{2^{-\mu-k}}^{2^{-k}} \varphi(F(x))\,dx + 2K_1 \sum_{k=1}^{N} \varphi(\omega_\varphi(2^{-k}, f)) + \sum_{k=2}^{n} 2^k N \varphi(\omega_\varphi(2^{-(k-1)N}, f))\right\}.$$

By (1. 15) it is clear that

$$\int_0^{2^{-(nN+1)}} \varphi(F(x))\,dx \leqq 2^{-n}\left\{2N \int_0^1 \varphi(F(x))\,dx + 2NK_1 \varphi(\omega_\varphi(\tfrac{1}{2}, f)) + \right. \tag{1.16}$$

$$\left. + N \sum_{k=2}^{n} 2^k \varphi(\omega_\varphi(2^{-(k-1)N}, f))\right\} \leqq$$

$$\leqq K_1(\varphi) 2^{-n}\left\{\int_0^1 \varphi(F(x))\,dx + \sum_{k=1}^{n-1} 2^k \varphi(\omega_\varphi(2^{-kN}, f))\right\}.$$

If $2^{nN+1} \leqq m < 2^{(n+1)N+1}$, then from (1. 16) it follows with $\varepsilon = (4[p+1]+2)^{-1} = \dfrac{1}{N}$ that

$$\int_0^{\frac{1}{m}} \varphi(F(x))\,dx \leqq \frac{K_2(\varphi)}{m^\varepsilon}\left\{\int_0^1 \varphi(F(x))\,dx + \sum_{k=1}^{\varepsilon \overset{2}{\smile}\!\log m} 2^k \varphi(\omega_\varphi(2^{-kN}, f))\right\}. \tag{1.17}$$

The estimation of the sum of the right-hand side is very easy. Indeed, we have

$$\sum_{k=1}^{\varepsilon \overset{2}{\smile}\!\log m} 2^k \varphi(\omega_\varphi(2^{-kN}, f)) \leqq 2^N \sum_{k=1}^{\varepsilon \overset{2}{\smile}\!\log m} \frac{2^k}{2^{kN}} \sum_{i=2^{(k-1)N}+1}^{2^{kN}} \varphi\left(\omega_\varphi\left(\frac{1}{i}, f\right)\right) \leqq$$

$$\leqq 2^{N+1} \sum_{k=1}^{\varepsilon \overset{2}{\smile}\!\log m} \sum_{i=2^{(k-1)N}+1}^{2^{kn}} i^{\varepsilon-1} \varphi\left(\omega_\varphi\left(\frac{1}{i}, f\right)\right) \leqq 2^{N+1} \sum_{i=1}^{m} i^{\varepsilon-1} \varphi\left(\omega_\varphi\left(\frac{1}{i}, f\right)\right);$$

inserting this into (1. 17), we obtain the statement of Lemma 9.

§ 2. Proof of the theorems

Proof of Theorem 1. Let $F(x)$ be the same function as in Lemma 1. Since for any nonnegative nondecreasing function $\chi(u)$ on $[0, \infty)$

$$\int_0^1 \chi(|f(x)|)\,dx = \int_0^1 \chi(F(x))\,dx$$

(see [6], p. 29), it is sufficient to prove (5) for $F(x)$.

Set

$$E_n = \{x : x \in [0,1], \quad n \leqq F(x) < n+1\}, \qquad (n = 0, 1, \dots).$$

It is clear that $E_n E_m = 0$ if $n \neq m$, $\sum_{n=0}^{\infty} E_n = [0,1]$, and if $E_n = \{\alpha_{n+1}, \alpha_n\}$ then $\alpha_n \downarrow 0$ as $n \to \infty$. Put

$$A_n = \int_0^{\alpha_n} \varphi(F(x))\,dx.$$

Then $A_n \to 0$ as $n \to \infty$, and thus for $n \geqq n_0$

$$\varphi(1) \geqq A_n \geqq \alpha_n \varphi(n) \geqq \alpha_n n \varphi(1),$$

that is $\alpha_n \leqq \dfrac{1}{n}$ for $n \geqq n_0$, and therefore

$$A_n \leqq \int_0^{\frac{1}{n}} \varphi(F(x))\,dx \qquad (n \geqq n_0). \tag{2.1}$$

To prove (5) we split the integral into an infinite sum and then make an Abel-transformation; thus we obtain

$$I \equiv \int_0^1 \varphi(F(x))\Lambda(F(x))\,dx = \sum_{n=0}^{\infty} \int_{E_n} \varphi(F(x))\,dx \sum_{k=1}^{n} \frac{\lambda_k}{k} \leqq$$

$$\leqq \sum_{k=1}^{\infty} \frac{\lambda_k}{k} \sum_{n=k}^{\infty} \int_{E_n} \varphi(F(x))\,dx = \sum_{k=1}^{\infty} \frac{\lambda_k}{k} A_k.$$

From this, (4), and (2.1), by using Lemma 9, we get that

$$I \leqq K(\varphi) \sum_{k=1}^{\infty} \frac{\lambda_k}{k^{1+\varepsilon}} \left\{ \sum_{n=1}^{k} n^{\varepsilon-1} \varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right) + \int_0^1 \varphi(F(x))\,dx \right\} \leqq$$

$$\leqq K(\varphi) \sum_{k=1}^{\infty} \frac{\lambda_k}{k^{1+\varepsilon}} \int_0^1 \varphi(F(x))\,dx + K(\varphi) \sum_{n=1}^{\infty} n^{\varepsilon-1} \varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right) \sum_{k=n}^{\infty} \frac{\lambda_k}{k^{1+\varepsilon}} \leqq$$

$$\leqq K(\varphi,\lambda) \int_0^1 \varphi(F(x))\,dx + K(\varphi,\lambda) \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \varphi\left(\omega_\varphi\left(\frac{1}{n}, f\right)\right),$$

whence (5) obviously follows.

The proof is thus complete.

Proof of Theorem 2. To prove (7) we use Lemma 4 and 7. It is clear that

$$I \equiv \int_0^1 \psi(|f(x)|)\varrho(|f(x)|)\,dx = \int_0^1 \psi(F(x))\varrho(F(x))\,dx = \tag{2.2}$$

$$= \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \psi(F(x))\varrho(F(x))\,dx \leqq \sum_{n=0}^{\infty} 2^{-n-1}\psi(F(2^{-n-1}))\varrho(F(2^{-n-1})).$$

Since, by (1), $\bar{\varphi}(x)x^{-1}$ decreases ($\bar{\varphi}(x)$ denotes the inverse of $\varphi(x)$), for any $c \geqq 1$ and $a \geqq 0$ we have

$$\bar{\varphi}(ca) \leqq c\bar{\varphi}(a).$$

This and (1. 7) imply

$$F(2^{-n}) \leqq K_1 2^n.$$

Using this inequality, (6) and (1. 7), by (2. 2) we have

$$I \leqq K_2 \sum_{n=0}^{\infty} 2^{-n}\varrho(2^n)\psi\Big(\int_0^1 \varphi(|f(x)|)\,dx + \sum_{k=1}^{n} \bar{\varphi}(2^k\varphi(\omega_\varphi(2^{-k},f))) \leqq$$

$$\leqq K_3 \sum_{n=1}^{\infty} 2^{-n}\varrho(2^n)\psi\Big(\int_0^1 \varphi(|f(x)|)\,dx\Big) +$$

$$+K_3 \sum_{n=0}^{\infty} 2^{-n}\varrho(2^n)\psi\left(\sum_{k=1}^{n} \bar{\varphi}(2^k\varphi(\omega_\varphi(2^{-k},f)))\right) \leqq K_4\psi\Big(\int_0^1 \varphi(|f(x)|)\,dx +$$

$$+K_5 \sum_{n=0}^{\infty} 2^{-n}\varrho(2^n)\psi\left(\sum_{k=1}^{n} \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{i}\bar{\varphi}\left[i\varphi\left[\omega_\varphi\left(\frac{1}{i},f\right)\right]\right]\right) \leqq \tag{2.3}$$

$$\leqq K_4\psi\Big(\int_0^1 \varphi(|f(x)|)\Big)\,dx +$$

$$+K_6 \sum_{m=1}^{\infty} \frac{\varrho(m)}{m^2}\psi\left(\sum_{i=1}^{m} \frac{1}{i}\bar{\varphi}\left[i\varphi\left[\omega_\varphi\left(\frac{1}{i},f\right)\right]\right]\right).$$

Applying Lemma 4, by (6), we obtain

$$\sum_{m=1}^{\infty} \frac{\varrho_m}{m^2}\psi\left(\sum_{i=1}^{m} \frac{1}{i}\bar{\varphi}\left[i\varphi\left[\omega_\varphi\left(\frac{1}{i},f\right)\right]\right]\right) \leqq K_6 \sum_{m=1}^{\infty} \frac{\varrho_m}{m^2}\psi\left(\bar{\varphi}\left[m\varphi\left[\omega_\varphi\left(\frac{1}{m},f\right)\right]\right]\right).$$

This and (2. 3) imply (7), and this completes the proof.

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(Received February 3, 1972)